MATH 321 Section 01
Homework 3

Below is a list of problems that I will collect Monday March 19. You should write up solutions carefully and neatly and ***staple your work***. All problems from the homework are fair game on the exams! You are encouraged to work in groups, but you must write up your own solutions. I will be available during office hours for help.

(1) Find the standard matrix for the linear operator defined by the formula.
   (a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$.
   (b) $K(x_1, x_2) = (x_1, x_2)$.
   (c) $E(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$.

(2) Let $P_n(\mathbb{R})$ denote the set of polynomials of degree at most n with real coefficients. Recall, the differential operator $D : P_n(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R})$ is a linear transformation. Find the corresponding matrix form for D in the case $n = 3$. [Hint: Represent x^3 as $(1, 0, 0, 0)$, x^2 as $(0, 1, 0, 0)$, x as $(0, 0, 1, 0)$, and 1 as $(0, 0, 0, 1)$.]

(3) Prove that the set M_{mn} (the set of all $m \times n$ real valued matrices) is a vector space under matrix addition and scalar multiplication.

(4) Let $A \in M_{mn}$, then we define the **kernel** of A to be the set $\ker A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$. Prove that $\ker A$ is a subspace of $\mathbb{R}^n$.

(5) Suppose

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 3 & -3 & 9 & 3 \\ 1 & 0 & 1 & -1 \\ 5 & -2 & 9 & -1 \end{pmatrix}.$$

Compute $\ker A$ and describe the set geometrically.

(6) Prove the Cauchy-Schwarz inequality $|u \cdot v| \leq \|u\|\|v\|$ for all vectors in $\mathbb{R}^n$.

(7) This exercise is to illustrate the concept of eigenvalues/eigenvectors. We start with a definition.

Definition 1. *Let $A \in M_{nn}$ and suppose that $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{R}^n$. We call λ an* ***eigenvalue*** *for A and $\vec{x}$ the* ***eigenvector*** *for A corresponding to λ.*

Show that λ is an eigenvalue for A with eigenvector $\vec{x}$ if and only if $\det(A - \lambda I) = 0$ where I is the $n \times n$ identity matrix.

(8) Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

Find all the eigenvalues for A as well as the corresponding eigenvectors.

(9) Let A be the diagonal matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where a and b are each positive real numbers.
   (a) Compute the eigenvalues λ_1, λ_2 and corresponding eigenvectors for A.
   (b) Show that the image of the unit circle by A is the ellipse with major axis length $\max\{\lambda_1, \lambda_2\}$ and minor axis length $\min\{\lambda_1, \lambda_2\}$.

(10) Recall that the matrix for a rotation by an angle θ about the origin is given by the matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

   (a) Show that for all $\theta \in (0, \pi)$ the matrix R_θ has no eigenvalues in $\mathbb{R}$.
   (b) Explain heuristically why this is the case.
   (c) Does the matrix R_π have any eigenvalues? What gives?

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