The Chain Rule

We now know how to take derivatives of several "elementary" functions, such as constant functions, identity functions, as well as the square root function. The next obstacle is to try to break down more complicated functions and express them in some way as a conglomeration of the functions we know about. There are many ways to do this. We can add functions together, multiply functions, divide one function into another, and compose functions together. We have seen how to deal with sums, products, and fractions of functions via the linear combination rule, the product rule, and the quotient rule respectively. The next natural question to ask is what do we do with compositions of functions? This is where the chain rule comes into play. The most important distinction about the chain rule (in my opinion) is that it is needed to take care of derivatives with respect to different *functions* as well as derivatives with respect to an independent variable. For example, if a man on the street runs at you with a function $f(u) = u^2$ and demands that you take it's derivative you would probably give him the derivative with respect to u. That is you would say that the derivative is $f'(u) = 2u$. But you need to be a little careful. What if he wanted the derivative with respect to a different variable t? Then there are two possible answers. If u is not a function of time, then it doesn't change as time changes right? So the rate of change of $f(u)$ with respect to t would be zero. For example, suppose that I have a function $A(r) = \pi r^2$ that tells me the area of a circle as a function of its radius. I can then ask what is the rate of change of area with respect to the radius. The answer would be $A'(r) = 2\pi r$. Now, if I ask what is the rate of change of the area of the circle with respect to the water level at Hoover dam you can see that there is no connection between these two quantities. If the water level at Hoover dam raises to 5,000 ft the change in the area of the circle will be zero since it depends on r. So, the rate of change of the area of the circle with respect to the water level in Hoover dam is identically zero.

Ok, now lets see if we can bump things up a notch. Suppose that the radius of the circle increases with respect to time. That is, suppose that the radius of the circle is described by some function of time $r(t)$. For example, to visualize this you can think of throwing a pebble into some water. At the time $t = 0$ the pebble has just touched the surface of the water and there really is not circle yet, i.e., $r(0) = 0$ and so $A(0) = \pi(r(0))^2 = 0$. Then as time moves forward, the radius of the circle of the ripple gets larger, thus the area of the circle gets larger. The key phrase in that last sentence was "as time moves forward." If time doesn't move forward then the radius does not change. Thus, the radius is dependent on time, and so the area is also dependent on time and we can talk about how the area changes with respect to time. But how would we compute such a rate? This is where we need the chain rule. The chain rule says that the rate of change of the area with respect to time will be the product of the rate of change of the area with respect to its radius and the rate of change of its radius with respect to time. That is,

$$\frac{d}{dt}A(r(t)) = \left(\frac{d}{dr}\pi(r(t))^2\right)\left(\frac{d}{dt}r(t)\right) = (2r(t))(r'(t)).$$

In general we have the following setup. If $h(x) = f(g(x))$ is a composition of differentiable functions, then the derivative of $h(x)$ with respect to x is

$$h'(x) = f'(g(x))g'(x).$$

That is, we pretend first that $g(x)$ is a variable and take a derivative with respect to that "variable", then take a derivative of $g(x)$ with respect to x as we know how to do. This is where I like to use the analogy of a virus trying to get at its host. First it has to break down the defenses and get to the heart before it can do any real damage. The derivative of a composition acts in a similar fashion. It needs to beat its way through the layers of a composition until it get to the independent variable. Let me elaborate with a deeper composition. Suppose $k(x) = f(g(h(x)))$. Then, how in the world would we compute the derivative of $k(x)$ with respect to x? Well, first we need to take care of the function on the outside. That function is $f(x)$. So we need to take a derivative of $f(x)$ with respect to the thing that it is eating, then we need to take a derivative of $g(x)$ with respect to the thing it is eating, and finally since $h(x)$ eats only the function x we see that after we compute the derivative of $h(x)$ with respect to the thing it eats we will be at the heart of the function. Indeed,

$$k'(x) = [f'(g(h(x))]\,[g'(h(x))]\,h'(x).$$

The hard part is identifying how to break down the composition. Let's look at a few examples.

1. Suppose $h(x) = (2x+1)^2$. Then we can see that we are squaring something right? That is we can write $h(x) = f(g(x))$ where $f(x) = x^2$ and $g(x) = 2x + 1$. So, if we want to take the derivative with respect to the function $g(x) = x + 1$ we see that we will get,

   $$\frac{d}{d(2x+1)}h(x) = 2(2x+1).$$

   That is, just pretend that $2x + 1$ is a variable and take a derivative with respect to that variable. Now, if we want to find the derivative of $h(x)$ with respect to x then we need to use the chain rule. I know that it must be $h'(x) = f'(g(x))g'(x)$ and we just computed $f'(g(x))$ above. Therefore since $g'(x) = 2$ we see that we have,

   $$\frac{d}{dx}h(x) = h'(x) = f'(g(x))g'(x) = [2(2x+1)](2).$$

2. Suppose that

$$k(x) = \frac{2}{(2x+1)^{37}}.$$

We need to identify how to decompose this beast as a composition of functions. It seems that the first layer looks like we are dividing 2 by a bunch of junk (where junk $= (2x+1)^{37}$). Notice that the function that looks like 2 divided by junk depends on more stuff than the $g(x) = x^{37}$ function. Specifically, the junk we are dividing 2 by is stuff raised to the 37^{th} power. That is we have,

$$k(x) = \frac{2}{(2x+1)^{37}} = \frac{2}{h(x)^{37}} = \frac{2}{g(h(x))} = f(g(h(x))).$$

Finally the stuff being raised to the 37^{th} power is exactly $h(x) = x + 1$ and this function only depends on x and so we can stop. Thus, we need to take the derivative of f with respect to $g(h(x))$, we need the derivative of g with respect to $h(x)$, then finally since $h(x)$ is only a function of x we can stop with the derivative of $h(x)$ with respect to x. So, let's compute. First pretend that $g(h(x))$ is a variable and compute the derivative of $k(x)$ with respect to this "variable."

$$f'(g(h(x))) = \frac{d}{dg(h(x))}\left(\frac{2}{(2x+1)^{37}}\right) = \frac{0(2x+1)^{37} - 2(1)}{((2x+1)^{37})^2} = -\frac{2}{((2x+1)^{37})^2}$$

That may have been a little hard to swallow. So there is another way to look at this that could be helpful. We want to pretend that $h(g(x))$ is a variable right? So, why not make it look like one. That is, let $u = h(g(x))$. Then we can make things look a little nicer by writing $f(g(h(x))) = f(u) = 2/u$ and compute the derivative,

$$f'(g(h(x))) = \frac{d}{dg(h(x))}f(g(h(x))) = \frac{d}{du}f(u) = \frac{d}{du}\frac{2}{u} = \frac{0(u) - 2(1)}{u^2} = -\frac{2}{u^2}.$$

Now, if we substitute back in we see that

$$-\frac{2}{u^2} = -\frac{2}{g(h(x))} = -\frac{2}{((2x+1)^{37})^2}.$$

Which is exactly what we got in the original computation.

Next we need to find $h'(g(x))$ so lets repeat what we did above. First the more detailed way,

$$g'(h(x)) = \frac{d}{dh(x)}g(h(x)) = 37(2x+1)^{36}.$$

Then, since we are pretending that $h(x) = 2x + 1$ is a variable we could say let $v = h(x)$. Then we would want to compute the derivative of g with respect to v. That is,

$$g'(h(x)) = \frac{d}{dh(x)} g(h(x)) = \frac{d}{dv} h(v) = \frac{d}{dv} v^{37} = 37v^{36}.$$

Next, substitution then gives us

$$g'(h(x)) = 37v^{36} = 37(h(x))^{3}6 = 37(2x+1)^{36}.$$

Which is what we got before.

Finally (*whew*) we only need to compute $h'(x)$, and since x is the independent variable we will have reached the heart and will be able to stop the chain of derivatives. So, $h'(x) = d/dx 2x + 1 = 2$. Thus we have all the information we need,

$$k'(x) = \frac{d}{dx} f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$$
$$= \left[-\frac{2}{\left((2x+1)^{37}\right)^2} \right] \left[37(2x+1)^{36}\right] [2]$$

One more thing. Lets see what this derivative would look like using the substituted variables.

$$\frac{dk}{dx} = \frac{df}{du} \frac{du}{dv} \frac{dv}{dx}$$

which looks like you are performing a change of units. This is where I got the analogy that I botched in class.

I sure hope this helps. If not, come by and see me and we will try to make it more clear.